

# Scattering of Spinning Test Particles by Plane Gravitational and Electromagnetic Waves

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## Abstract

The Mathisson-Papapetrou-Dixon (MPD) equations for the motion of electrically neutral massive spinning particles are analysed, in the pole-dipole approximation, in an Einstein-Maxwell plane-wave background spacetime. By exploiting the high symmetry of such spacetimes these equations are reduced to a system of tractable ordinary differential equations. Classes of exact solutions are given, corresponding to particular initial conditions for the directions of the particle spin relative to the direction of the propagating background fields. For Einstein-Maxwell pulses a scattering cross section is defined that reduces in certain limits to those associated with the scattering of scalar and Dirac particles based on classical and quantum field theoretic techniques. The relative simplicity of the MPD approach and its use of macroscopic spin distributions suggests that it may have advantages in those astrophysical situations that involve strong classical gravitational and electromagnetic environments.

## 1 Introduction

Einstein's theory of gravitation predicts that certain non-stationary distributions of matter will produce dynamic changes in the geometry of spacetime. Those that propagate are known as gravitational waves. Disturbances that also involve accelerating sources of the electromagnetic field are also expected to produce electromagnetic waves. Such scenarios can occur in many violent astrophysical processes and the energetics of such processes are poorly understood in many cases. It has also been suggested that matter with spin may play a dual role in many astrophysical processes that emit electromagnetic and gravitational waves. The constituents of plasmas that participate in the generation of such waves may become polarised in strong magnetic fields and spinning matter may respond to tidal forces that include interactions of spacetime curvature with angular momentum. Such processes belong to the realm of general relativistic transport theory. However some insights into the interaction of spinning test matter with gravitational and electromagnetic waves can be gained by studying the classical motion of spinning test particles in plane gravitational and electromagnetic field solutions to the Einstein-Maxwell equations. In this paper we explore the motion of massive electrically neutral particles with spin in such fields. Since the electromagnetic field influences the spacetime metric it will have an effect on such electrically neutral particles even in the absence of gravitational waves.

By neglecting self-gravitation and back-reaction the dynamics of classical test particles with angular momentum was first studied in detail by Mathisson, Fock, Papapetrou *et al* [1]. The theory was further clarified by Dixon [2, 3] using a rationalised multipole expansion technique and developed by Ehlers, Rudolph [4] and others. In this article we employ the pole-dipole approximation so that the motion of a classical spinning test particles is governed by the Mathisson-Papapetrou-Dixon (MPD) equations. Such MPD equations predict gravitational spin-spin interactions between rotating stars and orbiting massive spinning particles.

The properties of plane gravitational wave spacetimes have attracted considerable attention due to their high symmetry [6]. This has led to investigations of colliding gravitational and electromagnetic waves [7] and the scattering of massive particles in such backgrounds [8]. The scattering of scalar fields was considered by Gibbons [9], Garriga and Verdaguer [10] and others. The corresponding differential scattering cross-sections were found to be similar to

those for the scattering of classical spinless particles. The scattering of electrically neutral Dirac fields was subsequently investigated by Bini and Ferrari [11] and compared with that involving classical and quantum spinless particle scattering.

The classical scattering of neutral *spinning* particles by plane gravitational waves has been considered from a number of different perspectives [12, 13, 14]. In a recent paper [15] a special class of solutions of the MPD equations describing the non-geodesic motion of a massive spinning test particle in plane gravitational wave spacetime was constructed. The spin-curvature coupling was shown to give rise to parametric excitations of spinning matter by harmonic plane gravitational waves.

In this paper we generalise this analysis and suggest that information about the scattering of electrically neutral spinning matter in an Einstein-Maxwell plane wave background can be gleaned from solutions to the MPD equations by exploiting the high Killing symmetry of such spacetimes.

In section (2) the basic MPD equations are summarised and an expression given for constants of the motion that can be generated from Killing symmetries. Such expressions are pivotal in the subsequent analysis. In section (3) the MPD equations are expressed in different charts adapted to Einstein-Maxwell plane wave spacetimes. Such charts facilitate the symmetry reduction of the equations to a tractable system of ordinary differential equations. In section (4) several exact solutions to these equations are discussed. The final section exploits the properties of an Einstein-Maxwell pulse (sandwich spacetime) in order to estimate the scattering of an arbitrary distribution of classical spins according to the MPD equations. The result is compared with the classical and quantum scattering cross-sections of scalar and spinning particles calculated by other techniques in such backgrounds.

## 2 Equations of motion for neutral spinning test particles

In the monopole-dipole approximation the MPD equations determine the world-line of a spinning test particle with tangent vector  $V$ , momentum vector  $P$  and spin 2-form  $s$  in terms of the spacetime metric  $g$ , its Levi-Civita connection  $\nabla$  and Riemannian curvature tensor  $R$ .<sup>1</sup> By introducing the metric duals  $p \equiv \tilde{P} \equiv g(P, -)$  and  $v \equiv \tilde{V}$  and denoting interior operator (contraction with any vector  $V$ ) on forms by  $\iota_V$  [16], the MPD equations [3] can be written in a compact form as

$$\nabla_V p = \iota_V f \tag{1}$$

$$\nabla_V s = 2p \wedge v \tag{2}$$

with the spin condition [18]

$$\iota_P s = 0 \tag{3}$$

and forcing term

$$f = \frac{1}{4} \star^{-1} (R_{ab} \wedge \star s) e^a \wedge e^b \tag{4}$$

in terms of the exterior product  $\wedge$  and Hodge map  $\star$  associated with a metric with the signature  $(-, +, +, +)$ . Thus the “inverse Hodge map” denoted by  $\star^{-1}$  [16] acts on any  $q$ -form  $\omega$  according to  $\star^{-1}\omega = -(-1)^{q(4-q)} \star \omega$  satisfying  $\star(\star^{-1}\omega) = \star^{-1}(\star\omega) = \omega$ . The curvature 2-forms  $R^a_b$  in any basis  $\{X_a\}$  with co-basis  $\{e^a\}$  are related to the curvature tensor components  $R^a_{bcd}$  by  $2R^a_b(X_c, X_d) = R^a_{bcd}$ ,  $(a, b, c, d = 0, 1, 2, 3)$ . In terms of the components  $f_a = f(X_a)$ ,  $v^a = e^a(V)$ ,  $p^a = e^a(P)$ ,  $s_{ab} = s(X_a, X_b)$ , equation (4) becomes

$$f_a = -\frac{1}{2} R_{abcd} v^b s^{cd}.$$

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<sup>1</sup>Units in which  $c = G = 1$  are used throughout.

It follows that [5] the velocity of the spinning test particle is proportional to the tangent vector  $V$  whose components take the explicit form

$$v^a = \frac{p^c v_c}{p^f p_f} \left( p^a - \frac{2 R_{bcde} p^c s^{ab} s^{de}}{4 p^c p_c - R_{bcde} s^{bc} s^{de}} \right). \quad (5)$$

The freedom to normalise  $V$  ensures that  $p^c v_c$  is arbitrary and once a parameterisation of the world-line has been chosen (5) permits the computation of the world-line [4]. The norm of the momentum vector given by

$$m = \sqrt{-g(P, P)} \quad (6)$$

may be identified as the mass of the particle. This is used to define the normalised momentum vector given by

$$U = \frac{P}{m}. \quad (7)$$

In Minkowski spacetime  $U$  coincides with the 4-velocity  $V/\sqrt{-g(V, V)}$  but in general these two vectors differ in curved spacetime. Using the 1-form  $u = \tilde{U}$  the spin 1-form  $l$  is defined by  $l = -\frac{1}{2} \star (u \wedge s)$  and the spin vector by  $L = \tilde{l}$ . The angular-momentum 2-form  $s$  may be related back to the spin 1-form by

$$s = 2 \iota_U \star^{-1} l. \quad (8)$$

The norm of the spin vector  $L$  denoted  $\ell = \sqrt{g(L, L)}$  defines the “spin” of the particle. The reduced spin vector and 1-form are defined by  $\Sigma = L/m$  and  $\varsigma = l/m$  respectively such that the norm  $\alpha = \sqrt{g(\Sigma, \Sigma)} = \ell/m$  denotes the spin to mass ratio. It follows from (1), (2) and (3) that  $\alpha$  and  $m$  are constants of motion and the vectors  $U$  and  $\Sigma$  are orthogonal, i.e.  $g(U, \Sigma) = 0$ .

A vector field  $K$  on spacetime is a Killing vector field if

$$\mathcal{L}_K g = 0$$

$$\mathcal{L}_K \mathcal{F} = 0$$

for any Einstein-Maxwell solution  $\{g, \mathcal{F}\}$  satisfying

$$\mathcal{G}_a + 8\pi \mathcal{T}_a = 0 \quad (9)$$

$$d \star \mathcal{F} = 0 \quad (10)$$

where

$$\mathcal{T}_a = \frac{1}{2} (\iota_{X_a} \mathcal{F} \wedge \star \mathcal{F} - \iota_{X_a} \star \mathcal{F} \wedge \mathcal{F}) \quad (11)$$

in terms of the Einstein 3-forms  $\mathcal{G}_a$  in any basis  $\{X_a\}$  [16].

For each such  $K$  one may find a constant along the world-line  $\mathcal{C}$ . Sufficient constants of the motion enable one to integrate the equations of motion in terms of properties of the background metric. Given  $K = k^a X_a$  it follows from (1), (2) and (3) that the quantity

$$C_K \equiv \star^{-1} \left\{ \tilde{K} \wedge \star p + \frac{1}{4} d \tilde{K} \wedge \star s \right\} = k_a p^a - \frac{1}{2} k_{[a;b]} s^{ab} \quad (12)$$

is preserved along the world-line of a spinning test particle [3, 15]. Its value can therefore be fixed in terms of particular values of  $p$  and  $s$  at any event on their trajectories.

### 3 The MPD equations in an Einstein-Maxwell plane-wave background

The metric for an Einstein-Maxwell plane wave spacetime takes the form

$$g = H(\hat{\mu}, \hat{x}, \hat{y}) d\hat{\mu} \otimes d\hat{\mu} - \frac{1}{2} (d\hat{\mu} \otimes d\hat{\nu} + d\hat{\nu} \otimes d\hat{\mu}) + d\hat{x} \otimes d\hat{x} + d\hat{y} \otimes d\hat{y} \quad (13)$$

in “harmonic” (Kerr-Schild) coordinates  $(\hat{t}, \hat{x}, \hat{y}, \hat{z})$ . It describes a plane gravitational and electromagnetic wave travelling in the  $\hat{z}$ -direction [6, 7], where  $\hat{\mu} = \hat{t} - \hat{z}$ ,  $\hat{\nu} = \hat{t} + \hat{z}$  and

$$H(\hat{\mu}, \hat{x}, \hat{y}) = \phi_g(\hat{\mu}) (\hat{x}^2 - \hat{y}^2) + \phi_{\text{em}}(\hat{\mu}) (\hat{x}^2 + \hat{y}^2). \quad (14)$$

Here  $\phi_g(\hat{\mu})$  is an arbitrary gravitational wave profile whereas  $\phi_{\text{em}}(\hat{\mu})$  is related to the electromagnetic potential 1-form

$$\mathcal{A} = \mathcal{A}_1(\hat{\mu}) d\hat{x} + \mathcal{A}_2(\hat{\mu}) d\hat{y} \quad (15)$$

with arbitrary waveforms  $\mathcal{A}_1(\hat{\mu})$  and  $\mathcal{A}_2(\hat{\mu})$  by

$$\phi_{\text{em}}(\hat{\mu}) = -2\pi (\mathcal{A}'_1(\hat{\mu})^2 + \mathcal{A}'_2(\hat{\mu})^2) \quad (16)$$

and  $'$  denotes differentiation. The electromagnetic field 2-form  $\mathcal{F} = d\mathcal{A}$ .

In the orthonormal co-basis defined by

$$\begin{aligned} e^0 &= \frac{d\hat{\nu}}{2} + (1 - H(\hat{\mu}, \hat{x}, \hat{y})) \frac{d\hat{\mu}}{2} \\ e^1 &= d\hat{x}, \quad e^2 = d\hat{y} \\ e^3 &= \frac{d\hat{\nu}}{2} - (1 + H(\hat{\mu}, \hat{x}, \hat{y})) \frac{d\hat{\mu}}{2} \end{aligned} \quad (17)$$

let  $\{X_a\}$  be the dual basis satisfying  $e^a(X_b) = \delta^a_b$ . This basis is parallel along the geodesic observer  $\mathcal{O} : \tau \mapsto (\hat{t}(\tau) = \tau, \hat{x} = 0, \hat{y} = 0, \hat{z} = 0)$ , i.e.  $\nabla X_a|_{\mathcal{O}} = 0$ , and takes the form  $\{X_0 = \partial_{\hat{t}}, X_1 = \partial_{\hat{x}}, X_2 = \partial_{\hat{y}}, X_3 = \partial_{\hat{z}}\}$  and sets up a “local Lorentz frame” along  $\mathcal{O}$ . In these coordinates this observer curve is parameterised by  $\mathcal{O} : \tau \mapsto (\hat{\mu}(\tau) = \tau, \hat{\nu}(\tau) = \tau, \hat{x} = 0, \hat{y} = 0)$ .

We are interested in the motion of a spinning test particle with mass  $m$ , spin  $\ell = \alpha m$  initially at “rest” relative to observer  $\mathcal{O}$  and excited by a pulse of plane gravitational and electromagnetic waves. Thus we parameterise the world-line of this particle by  $\mathcal{C} : \lambda \mapsto (\hat{\mu}(\lambda) = \lambda, \hat{\nu}(\lambda), \hat{x}(\lambda), \hat{y}(\lambda))$  for functions  $\hat{\nu}(\lambda), \hat{x}(\lambda), \hat{y}(\lambda)$  subject to the initial conditions:

$$\hat{\nu}(0) = 0, \quad \hat{x}(0) = x_{\text{in}}, \quad \hat{y}(0) = y_{\text{in}} \quad (18)$$

with arbitrary position parameters  $x_{\text{in}}$  and  $y_{\text{in}}$ . The tangent vector  $V$  to the worldline  $\mathcal{C}$  is written in these coordinates:

$$V = C' = \partial_{\hat{\mu}} + \hat{\nu}'(\lambda) \partial_{\hat{\nu}} + \hat{x}'(\lambda) \partial_{\hat{x}} + \hat{y}'(\lambda) \partial_{\hat{y}}.$$

The pulse nature of the wave can be accommodated by taking the arbitrary profiles in the Einstein-Maxwell solution above to have compact support in the variable  $\hat{\mu}$ . Where such profiles vanish the spacetime is flat.

Since the mass  $m$  and spin  $\alpha$  of the test particles are constants of the motion one has

$$\eta_{ab} u^a(\lambda) u^b(\lambda) = -1 \quad (19)$$

$$\eta_{ab} \varsigma^a(\lambda) \varsigma^b(\lambda) = \alpha^2 \quad (20)$$

$$\eta_{ab} u^a(\lambda) \varsigma^b(\lambda) = 0 \quad (21)$$

in terms of the orthonormal components  $u^a = e^a(U)$  and  $\varsigma^a = e^a(\Sigma)$  with  $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ . The initial conditions for  $u^a$  are:

$$u^0(0) = 1, \quad u^1(0) = u^2(0) = u^3(0) = 0 \quad (22)$$

and we write the initial conditions for  $\varsigma^a$

$$\varsigma^0(0) = 0, \quad \varsigma^1(0) = \varsigma_{\text{in}}^1, \quad \varsigma^2(0) = \varsigma_{\text{in}}^2, \quad \varsigma^3(0) = \varsigma_{\text{in}}^3 \quad (23)$$

where the constants  $\varsigma_{\text{in}}^1, \varsigma_{\text{in}}^2, \varsigma_{\text{in}}^3$  satisfy  $(\varsigma_{\text{in}}^1)^2 + (\varsigma_{\text{in}}^2)^2 + (\varsigma_{\text{in}}^3)^2 = \alpha^2$ .

For convenience let

$$\phi_{\pm}(\hat{\mu}) \equiv \phi_{\text{g}}(\hat{\mu}) \pm \phi_{\text{em}}(\hat{\mu}) \quad (24)$$

$$\Phi_{\pm}(\hat{\mu}) \equiv \int_0^{\hat{\mu}} \phi_{\pm}(\zeta) \, d\zeta. \quad (25)$$

The Killing symmetry of the Einstein-Maxwell plane wave spacetime is most conveniently expressed in “group” (Rosen) coordinates  $(\mu, \nu, x, y)$ . For suitable ranges these are related to the previous chart by the relations:

$$\hat{\mu} = \mu \quad (26)$$

$$\hat{\nu} = \nu + x^2 a(\mu) a'(\mu) + y^2 b(\mu) b'(\mu) \quad (27)$$

$$\hat{x} = a(\mu) x \quad (28)$$

$$\hat{y} = b(\mu) y \quad (29)$$

where the metric functions  $a(\mu)$  and  $b(\mu)$  satisfy

$$a''(\mu) = \phi_+(\mu) a(\mu) \quad (30)$$

$$b''(\mu) = -\phi_-(\mu) b(\mu) \quad (31)$$

with a convenient choice of the (gauge fixing) conditions:

$$a(0) = b(0) = 1 \quad (32)$$

$$a'(0) = b'(0) = 0. \quad (33)$$

In this coordinate chart the metric  $g$  takes the form

$$g = -\frac{1}{2} (d\mu \otimes d\nu + d\nu \otimes d\mu) + a(\mu)^2 dx \otimes dx + b(\mu)^2 dy \otimes dy. \quad (34)$$

This Einstein-Maxwell spacetime admits five independent Killing vectors [7]:

$$K_1 = \frac{\partial}{\partial \nu}, \quad K_2 = \frac{\partial}{\partial x}, \quad K_3 = \frac{\partial}{\partial y} \quad (35)$$

$$K_4 = A(\mu) \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial \nu} \quad (36)$$

$$K_5 = B(\mu) \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial \nu} \quad (37)$$

where

$$A(\mu) = \int_0^\mu \frac{d\zeta}{a(\zeta)^2}, \quad B(\mu) = \int_0^\mu \frac{d\zeta}{b(\zeta)^2}. \quad (38)$$

In these coordinates we write the tangent vector

$$V = \partial_\mu + \nu'(\lambda)\partial_\nu + x'(\lambda)\partial_x + y'(\lambda)\partial_y.$$

It follows from (12), (18), (22) and (23) that

$$C_{K_1} = -\frac{m}{2}, \quad C_{K_2} = 0, \quad C_{K_3} = 0, \quad C_{K_4} = m\beta_2, \quad C_{K_5} = -m\beta_1 \quad (39)$$

where  $\beta_1 = \varsigma_{\text{in}}^1 + y_{\text{in}}$  and  $\beta_2 = \varsigma_{\text{in}}^2 - x_{\text{in}}$ . From (18) and (26) – (29) in group coordinates the initial conditions for the world-line of the spinning test particle  $\mathcal{C} : \lambda \mapsto (\mu(\lambda) = \lambda, \nu(\lambda), x(\lambda), y(\lambda))$  read

$$\nu(0) = 0, \quad x(0) = x_{\text{in}}, \quad y(0) = y_{\text{in}}. \quad (40)$$

The five Killing vectors above yield five algebraic relations, (12), between the components of  $u$  and  $\varsigma$  for any  $\lambda$ . Equations (19), (20), (21) provide three further algebraic relations. These eight equations may be solved in terms of the constants of motion and initial conditions to yield the solutions (41) and (42) below:

$$\begin{aligned} u^0(\lambda) &= 1 + \frac{\beta_2^2}{2} a'(\lambda)^2 + \frac{\beta_1^2}{2} b'(\lambda)^2 \\ u^1(\lambda) &= -\beta_2 a'(\lambda), \quad u^2(\lambda) = \beta_1 b'(\lambda) \\ u^3(\lambda) &= \frac{\beta_2^2}{2} a'(\lambda)^2 + \frac{\beta_1^2}{2} b'(\lambda)^2 \end{aligned} \quad (41)$$

and

$$\begin{aligned} \varsigma^0(\lambda) &= \frac{1}{\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2} \left\{ (\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2) \varsigma^3(\lambda) \right. \\ &\quad \left. - 2(y(\lambda) - \beta_1)\beta_2 b(\lambda)a'(\lambda) - 2(x(\lambda) + \beta_2)\beta_1 a(\lambda)b'(\lambda) \right\} \\ \varsigma^1(\lambda) &= -(y(\lambda) - \beta_1)b(\lambda) + \frac{1}{\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2} \left\{ -2\beta_2 a'(\lambda)\varsigma^3(\lambda) \right. \\ &\quad \left. + 2\beta_2 a'(\lambda) [(y(\lambda) - \beta_1)\beta_2 b(\lambda)a'(\lambda) + (x(\lambda) + \beta_2)\beta_1 a(\lambda)b'(\lambda)] \right\} \\ \varsigma^2(\lambda) &= (x(\lambda) + \beta_2)a(\lambda) + \frac{1}{\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2} \left\{ 2\beta_1 b'(\lambda)\varsigma^3(\lambda) \right. \\ &\quad \left. - 2\beta_1 b'(\lambda) [(y(\lambda) - \beta_1)\beta_2 b(\lambda)a'(\lambda) + (x(\lambda) + \beta_2)\beta_1 a(\lambda)b'(\lambda)] \right\}. \end{aligned} \quad (42)$$

Furthermore they imply a quadratic equation for  $\varsigma^3(\lambda)$ :

$$\varsigma^3(\lambda)^2 + \mathcal{P}(\lambda) \varsigma^3(\lambda) + \mathcal{Q}(\lambda) = 0 \quad (43)$$

where

$$\mathcal{P}(\lambda) = -2(y(\lambda) - \beta_1)\beta_2 b(\lambda)a'(\lambda) - 2(x(\lambda) + \beta_2)\beta_1 a(\lambda)b'(\lambda) \quad (44)$$

and

$$\begin{aligned}
\mathcal{Q}(\lambda) = & -\frac{\beta^2}{4} \{ \beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2 \}^2 \\
& + \frac{(y(\lambda) - \beta_1)^2 b(\lambda)^2}{4} \{ 2\beta_1^2 \beta_2^2 a'(\lambda)^2 b'(\lambda)^2 + \beta_2^4 a'(\lambda)^4 + \beta_1^4 b'(\lambda)^4 - 4\beta_1^2 b'(\lambda)^2 + 4 \} \\
& + \frac{(x(\lambda) + \beta_2)^2 a(\lambda)^2}{4} \{ 2\beta_1^2 \beta_2^2 a'(\lambda)^2 b'(\lambda)^2 + \beta_2^4 a'(\lambda)^4 + \beta_1^4 b'(\lambda)^4 - 4\beta_2^2 a'(\lambda)^2 + 4 \} \\
& + 2\beta_1 \beta_2 a(\lambda) b(\lambda) a'(\lambda) b'(\lambda) (x(\lambda) + \beta_2)(y(\lambda) - \beta_1).
\end{aligned} \tag{45}$$

Thus for given metric functions  $a(\lambda)$ ,  $b(\lambda)$  one has solutions  $\varsigma^3(\lambda)$  in terms of  $x(\lambda)$  and  $y(\lambda)$ .

The differential equations for the components  $\nu(\lambda)$ ,  $x(\lambda)$ ,  $y(\lambda)$  of the tangent vector  $V$  are obtained from substituting (7), (8), (41) and (42) into (5):

$$\begin{aligned}
x'(\lambda) = & 2\phi_-(\lambda) \{ \beta_1 a(\lambda) b(\lambda) b'(\lambda) (x(\lambda) + \beta_2)(y(\lambda) - \beta_1) + \beta_2 b(\lambda)^2 a'(\lambda) (y(\lambda) - \beta_1)^2 \\
& - (y(\lambda) - \beta_1) b(\lambda) \varsigma^3(\lambda) \} / \{ a(\lambda) (\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2) \} - \frac{(x(\lambda) + \beta_2) a'(\lambda)}{a(\lambda)}
\end{aligned} \tag{46}$$

$$\begin{aligned}
y'(\lambda) = & 2\phi_+(\lambda) \{ \beta_2 a(\lambda) b(\lambda) a'(\lambda) (x(\lambda) + \beta_2)(y(\lambda) - \beta_1) + \beta_1 a(\lambda)^2 b'(\lambda) (x(\lambda) + \beta_2)^2 \\
& - (x(\lambda) + \beta_2) a(\lambda) \varsigma^3(\lambda) \} / \{ b(\lambda) (\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2) \} - \frac{(y(\lambda) - \beta_2) b'(\lambda)}{b(\lambda)}
\end{aligned} \tag{47}$$

$$\begin{aligned}
\nu'(\lambda) = & 1 + (x(\lambda) + \beta_2)^2 a'(\lambda)^2 + (y(\lambda) - \beta_1)^2 b'(\lambda)^2 \\
& + \frac{\phi_+(\lambda)}{\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2} \left\{ -2a(\lambda)^2 (x(\lambda) + \beta_2)^2 (\beta_2^2 a'(\lambda)^2 + 2\beta_1 b'(\lambda)^2 y(\lambda) - \beta_1^2 b'(\lambda)^2 - 2) \right. \\
& \left. - 4\beta_2 a(\lambda) b(\lambda) a'(\lambda) b'(\lambda) (x(\lambda) + \beta_2) (y(\lambda) - \beta_1)^2 + 4a(\lambda) b'(\lambda) \varsigma^3(\lambda) (x(\lambda) + \beta_2) (y(\lambda) - \beta_1) \right\} \\
& + \frac{\phi_-(\lambda)}{\beta_2^2 a'(\lambda)^2 + \beta_1^2 b'(\lambda)^2 - 2} \left\{ 2b(\lambda)^2 (y(\lambda) - \beta_1)^2 (\beta_1^2 b'(\lambda)^2 - 2\beta_2 a'(\lambda)^2 x(\lambda) - \beta_2^2 a'(\lambda)^2 - 2) \right. \\
& \left. - 4\beta_1 a(\lambda) b(\lambda) a'(\lambda) b'(\lambda) (x(\lambda) + \beta_2)^2 (y(\lambda) - \beta_1) + 4b(\lambda) a'(\lambda) \varsigma^3(\lambda) (y(\lambda) - \beta_1) (x(\lambda) + \beta_2) \right\}
\end{aligned} \tag{48}$$

These equations are decoupled as the sets:

$$\begin{pmatrix} x(\lambda) \\ y(\lambda) \end{pmatrix}' = \begin{pmatrix} F_1(x(\lambda), y(\lambda), \varsigma^3(\lambda)) \\ F_2(x(\lambda), y(\lambda), \varsigma^3(\lambda)) \end{pmatrix} \tag{49}$$

and

$$\nu'(\lambda) = F_3(x(\lambda), y(\lambda), \varsigma^3(\lambda)) \tag{50}$$

where from (43),

$$\varsigma^3(\lambda) = F_4(x(\lambda), y(\lambda)).$$

Thus the world-line of the spinning particle can be determined by solving the two coupled non-linear differential equations (49) for  $x(\lambda)$  and  $y(\lambda)$  and inserting these solutions into (50) and solving for  $\nu(\lambda)$ . The evolution of the reduced spin vector  $\Sigma$  then follows by solving (42) and (43).

Given the initial position parameters  $x_{\text{in}}, y_{\text{in}}$  and reduced spin vector components  $\varsigma_{\text{in}}^1, \varsigma_{\text{in}}^2, \varsigma_{\text{in}}^3$  the evolution of the normalised momentum vector  $U$  follows more simply from (41) for arbitrary profiles  $\phi_{\pm}(\mu)$  and the corresponding  $a(\mu), b(\mu)$  obtained from (30), (31), (32), (33).

## 4 Particular solutions describing geodesic and non-geodesic motions

Since the above equations are non-linear it is difficult to find general analytic solutions for the particle's worldline and spin. It is however possible to find interesting particular analytic solutions for special initial conditions. Thus if the initial spin vector is in the direction of propagation of the gravitational and/or electromagnetic wave then it remains so and furthermore the subsequent motion is geodesic. With  $\varsigma_{\text{in}}^1 = \varsigma_{\text{in}}^2 = 0, \alpha = |\varsigma_{\text{in}}^3|$  and any  $x_{\text{in}}$  and  $y_{\text{in}}$ , the world-line  $\mathcal{C}$  is simply:

$$\mu(\lambda) = \lambda, \quad x(\lambda) = x_{\text{in}}, \quad y(\lambda) = y_{\text{in}}, \quad \nu(\lambda) = \lambda \quad (51)$$

with normalised momentum components:

$$\begin{aligned} u^0(\lambda) &= 1 + \frac{x_{\text{in}}^2}{2} a'(\lambda)^2 + \frac{y_{\text{in}}^2}{2} b'(\lambda)^2 \\ u^1(\lambda) &= x_{\text{in}} a'(\lambda), \quad u^2(\lambda) = y_{\text{in}} b'(\lambda) \\ u^3(\lambda) &= \frac{x_{\text{in}}^2}{2} a'(\lambda)^2 + \frac{y_{\text{in}}^2}{2} b'(\lambda)^2 \end{aligned} \quad (52)$$

and reduced spin components:

$$\begin{aligned} \varsigma^0(\lambda) &= -\frac{\varsigma_{\text{in}}^3}{2} (x_{\text{in}}^2 a'(\lambda)^2 + y_{\text{in}}^2 b'(\lambda)^2) \\ \varsigma^1(\lambda) &= -\varsigma_{\text{in}}^3 x_{\text{in}} a'(\lambda), \quad \varsigma^2(\lambda) = -\varsigma_{\text{in}}^3 y_{\text{in}} b'(\lambda) \\ \varsigma^3(\lambda) &= \frac{\varsigma_{\text{in}}^3}{2} (1 - x_{\text{in}}^2 a'(\lambda)^2 - y_{\text{in}}^2 b'(\lambda)^2). \end{aligned} \quad (53)$$

A class of solutions describing *non-geodesic* motions can also be found in which the particle spin vector is transported  $\nabla$ -parallel along its world-line, transverse to the direction of wave propagation. Consider the case with  $\alpha = |\varsigma_{\text{in}}^1|, \varsigma_{\text{in}}^2 = \varsigma_{\text{in}}^3 = 0$  and

$$x(\lambda) = x_{\text{in}}. \quad (54)$$

Then (46) implies

$$\varsigma^3(\lambda) = x_{\text{in}} (\varsigma_{\text{in}}^1 + y_{\text{in}} - y(\lambda)) b(\lambda) a'(\lambda) \quad (55)$$

and (47) becomes

$$y'(\lambda) = \frac{(\varsigma_{\text{in}}^1 + y_{\text{in}} - y(\lambda)) b'(\lambda)}{b(\lambda)} \quad (56)$$

with the solution

$$y(\lambda) = \varsigma_{\text{in}}^1 \left( 1 - \frac{1}{b(\lambda)} \right) + y_{\text{in}} \quad (57)$$



satisfying the initial condition  $y(0) = y_{\text{in}}$ . Equation (48) then reduces to

$$\nu'(\lambda) = 1 + (\varsigma_{\text{in}}^1)^2 \left( 2\phi_-(\lambda) + \frac{b'(\lambda)^2}{b(\lambda)^2} \right). \quad (58)$$

Using (31) and the initial condition  $\nu(0) = 0$  this can be integrated :

$$\nu(\lambda) = \lambda + (\varsigma_{\text{in}}^1)^2 \left( \Phi_-(\lambda) - \frac{b'(\lambda)}{b(\lambda)} \right). \quad (59)$$

Substituting the above equations into (41) and (42) gives

$$\begin{aligned} u^0(\lambda) &= 1 + \frac{x_{\text{in}}^2}{2} a'(\lambda)^2 + \frac{(\varsigma_{\text{in}}^1 + y_{\text{in}})^2}{2} b'(\lambda)^2 \\ u^1(\lambda) &= x_{\text{in}} a'(\lambda), \quad u^2(\lambda) = (\varsigma_{\text{in}}^1 + y_{\text{in}}) b'(\lambda) \\ u^3(\lambda) &= \frac{x_{\text{in}}^2}{2} a'(\lambda)^2 + \frac{(\varsigma_{\text{in}}^1 + y_{\text{in}})^2}{2} b'(\lambda)^2 \end{aligned} \quad (60)$$

and

$$\varsigma^0(\lambda) = \varsigma_{\text{in}}^1 x_{\text{in}} a'(\lambda), \quad \varsigma^1(\lambda) = \varsigma_{\text{in}}^1, \quad \varsigma^2(\lambda) = 0, \quad \varsigma^3(\lambda) = \varsigma_{\text{in}}^1 x_{\text{in}} a'(\lambda). \quad (61)$$

Using (26) – (29) solutions (54), (57), (59) in harmonic coordinates become

$$\begin{aligned} \hat{t}(\lambda) &= \lambda + \frac{x_{\text{in}}^2}{2} a(\lambda) a'(\lambda) + \frac{(\varsigma_{\text{in}}^1 + y_{\text{in}})^2}{2} b(\lambda) b'(\lambda) - 2\varsigma_{\text{in}}^1 (\varsigma_{\text{in}}^1 + y_{\text{in}}) b'(\lambda) + \frac{(\varsigma_{\text{in}}^1)^2}{2} \Phi_-(\lambda) \\ \hat{x}(\lambda) &= a(\lambda) x_{\text{in}}, \quad \hat{y}(\lambda) = b(\lambda) (\varsigma_{\text{in}}^1 + y_{\text{in}}) - \varsigma_{\text{in}}^1 \\ \hat{z}(\lambda) &= \frac{x_{\text{in}}^2}{2} a(\lambda) a'(\lambda) + \frac{(\varsigma_{\text{in}}^1 + y_{\text{in}})^2}{2} b(\lambda) b'(\lambda) - 2\varsigma_{\text{in}}^1 (\varsigma_{\text{in}}^1 + y_{\text{in}}) b'(\lambda) + \frac{(\varsigma_{\text{in}}^1)^2}{2} \Phi_-(\lambda). \end{aligned} \quad (62)$$

For such non-geodesic motion the difference between the corresponding momentum and velocity vectors is nonzero and takes the form:

$$U - V = -2 (\varsigma_{\text{in}}^1)^2 \phi_-(\lambda) \frac{\partial}{\partial \hat{\nu}}. \quad (63)$$

With the alternative initial spin conditions  $\varsigma_{\text{in}}^1 = \varsigma_{\text{in}}^3 = 0$  and  $\alpha = |\varsigma_{\text{in}}^2|$ , one similarly finds:

$$\begin{aligned} u^0(\lambda) &= 1 + \frac{(\varsigma_{\text{in}}^2 - x_{\text{in}})^2}{2} a'(\lambda)^2 + \frac{y_{\text{in}}^2}{2} b'(\lambda)^2, \\ u^1(\lambda) &= -(\varsigma_{\text{in}}^2 - x_{\text{in}}) a'(\lambda), \quad u^2(\lambda) = y_{\text{in}} b'(\lambda) \\ u^3(\lambda) &= \frac{(\varsigma_{\text{in}}^2 - x_{\text{in}})^2}{2} a'(\lambda)^2 + \frac{y_{\text{in}}^2}{2} b'(\lambda)^2 \end{aligned} \quad (64)$$

and

$$\varsigma^0(\lambda) = \varsigma_{\text{in}}^2 y_{\text{in}} b'(\lambda), \quad \varsigma^1(\lambda) = 0, \quad \varsigma^2(\lambda) = \varsigma_{\text{in}}^2, \quad \varsigma^3(\lambda) = \varsigma_{\text{in}}^2 y_{\text{in}} b'(\lambda). \quad (65)$$

The corresponding world-line is then given by

$$\hat{t}(\lambda) = \lambda \frac{(\varsigma_{\text{in}}^2 - x_{\text{in}})^2}{2} a(\lambda) a'(\lambda) + \frac{y_{\text{in}}^2}{2} b(\lambda) b'(\lambda) - 2 \varsigma_{\text{in}}^2 (\varsigma_{\text{in}}^2 - x_{\text{in}}) a'(\lambda) - \frac{(\varsigma_{\text{in}}^2)^2}{2} \Phi_+(\lambda) \quad (66)$$

$$\hat{x}(\lambda) = -a(\lambda)(\varsigma_{\text{in}}^2 - x_{\text{in}}) + \varsigma_{\text{in}}^2 \quad (67)$$

$$\hat{y}(\lambda) = b(\lambda) y_{\text{in}} \quad (68)$$

$$\hat{z}(\lambda) = \frac{(\varsigma_{\text{in}}^2 - x_{\text{in}})^2}{2} a(\lambda) a'(\lambda) + \frac{y_{\text{in}}^2}{2} b(\lambda) b'(\lambda) - 2 \varsigma_{\text{in}}^2 (\varsigma_{\text{in}}^2 - x_{\text{in}}) a'(\lambda) - \frac{(\varsigma_{\text{in}}^2)^2}{2} \Phi_+(\lambda) \quad (69)$$

in harmonic coordinates. In this case the difference between momentum and velocity vectors is:

$$U - V = 2 (\varsigma_{\text{in}}^2)^2 \phi_+(\lambda) \frac{\partial}{\partial \nu}. \quad (70)$$

## 5 Scattering cross-section due to an Einstein-Maxwell wave pulse

Aside from regularity conditions the above calculations have not imposed any strong conditions on the structure of the profile functions that characterise the Einstein-Maxwell background spacetime. Consider now the scattering of neutral massive spinning particles by a plane gravitational and electromagnetic wave pulse represented by the metric (13) with functions  $\phi_g(\hat{\mu})$  and  $\phi_{\text{em}}(\hat{\mu})$  that vanish for  $\hat{\mu} < 0$  and  $\hat{\mu} > \delta$ , where  $\delta$  denotes the width of the wave pulse. Such a sandwich wave gives rise to two half Minkowski spacetimes namely  $\mathcal{M}_{\text{in}}$  for  $\hat{\mu} < 0$  and  $\mathcal{M}_{\text{out}}$  for  $\hat{\mu} > \delta$  corresponding to the domains before and after the passage of the plane wave pulse. Within each half Minkowski spacetime the orthonormal basis  $\{X_0 = \partial_{\hat{t}}, X_1 = \partial_{\hat{x}}, X_2 = \partial_{\hat{y}}, X_3 = \partial_{\hat{z}}\}$  defines a Lorentz frame, and  $\{X_a\}$  along the geodesic observer  $\mathcal{O}$  still remains a local Lorentz basis also in the “sandwiched” domain  $\mathcal{D}$  where  $0 \leq \hat{\mu} = \hat{t} - \hat{z} \leq \delta$ . Consider a collection of spinning particles, labelled by their initial  $\hat{x}$  and  $\hat{y}$  components, i.e.  $x_{\text{in}}$  and  $y_{\text{in}}$ . Suppose these particles are initially at rest relative to  $\mathcal{O}$  having common initial velocity and momentum given by  $V(0) = U(0) = X_0$  but with possibly non-uniform initial spin orientations described by:

$$\varsigma_{\text{in}}^1 = \varsigma_{\text{in}}^1(x_{\text{in}}, y_{\text{in}}), \quad \varsigma_{\text{in}}^2 = \varsigma_{\text{in}}^2(x_{\text{in}}, y_{\text{in}}), \quad \varsigma_{\text{in}}^3 = \varsigma_{\text{in}}^3(x_{\text{in}}, y_{\text{in}}). \quad (71)$$

Particles scattered by the pulse will have different out-going momentum vectors, depending on their initial data. In  $\mathcal{M}_{\text{in}}$  and  $\mathcal{M}_{\text{out}}$  these vectors have different constant components  $u^a$  and changes in these quantities along  $\mathcal{C}$  across the domain  $\mathcal{D}$  reflect their gain in energy-momentum in the Lorentz frame associated with  $\mathcal{O}$ . In any chart for  $\mathcal{M}_{\text{out}}$  write  $a'_{\text{out}} \equiv a'(\lambda)$ ,  $b'_{\text{out}} \equiv b'(\lambda)$  and  $p_{\text{out}}^a \equiv m u^a(\lambda)$  for any  $\lambda > \delta$ . In such a sandwich spacetime the components of the final momentum vector are readily evaluated from (41):

$$p_{\text{out}}^0 = m + \frac{m}{2} (\varsigma_{\text{in}}^2 - x_{\text{in}})^2 a'_{\text{out}}{}^2 + \frac{m}{2} (\varsigma_{\text{in}}^1 + y_{\text{in}})^2 b'_{\text{out}}{}^2 \quad (72)$$

$$p_{\text{out}}^1 = -m (\varsigma_{\text{in}}^2 - x_{\text{in}}) a'_{\text{out}} \quad (73)$$

$$p_{\text{out}}^2 = m (\varsigma_{\text{in}}^1 + y_{\text{in}}) b'_{\text{out}} \quad (74)$$

$$p_{\text{out}}^3 = \frac{m}{2} (\varsigma_{\text{in}}^2 - x_{\text{in}})^2 a'_{\text{out}}{}^2 + \frac{m}{2} (\varsigma_{\text{in}}^1 + y_{\text{in}})^2 b'_{\text{out}}{}^2. \quad (75)$$

Equations (73) and (74) define a map between the transverse components of momentum in any spacelike plane associated with  $\mathcal{O}$ 's rest space in  $\mathcal{M}_{\text{out}}$  and the initial location of particles in a similar plane in  $\mathcal{M}_{\text{in}}$ . It is therefor natural to define a classical differential scattering cross-section associated with this map in terms of the out-going momentum components  $(p_{\text{out}}^1, p_{\text{out}}^2)$ :

$$d\sigma \equiv dx_{\text{in}} dy_{\text{in}} = \frac{dp_{\text{out}}^1 dp_{\text{out}}^2}{m^2 a'_{\text{out}} b'_{\text{out}}} \left( 1 + \frac{\partial \varsigma_{\text{in}}^1}{\partial y_{\text{in}}} - \frac{\partial \varsigma_{\text{in}}^2}{\partial x_{\text{in}}} + \frac{\partial \varsigma_{\text{in}}^1}{\partial x_{\text{in}}} \frac{\partial \varsigma_{\text{in}}^2}{\partial y_{\text{in}}} - \frac{\partial \varsigma_{\text{in}}^1}{\partial y_{\text{in}}} \frac{\partial \varsigma_{\text{in}}^2}{\partial x_{\text{in}}} \right)^{-1}. \quad (76)$$

This cross-section provides a means of estimating the effects of a non-uniform distribution of spin on the scattering of particles by a passing Einstein-Maxwell pulse of radiation. When this initial polarisation is uniform (i.e.  $\varsigma_{\text{in}}^1, \varsigma_{\text{in}}^2$  are independent of the parameters  $x_{\text{in}}, y_{\text{in}}$ ) this expression agrees with the differential cross-section describing the scattering of classical or quantum non-spinning particles [10] by an Einstein-Maxwell pulse and furthermore coincides with the differential cross-section describing the scattering of *polarised* incident quantum Dirac particles in such a background [11]. A priori there is no reason to expect that electrically neutral particles with spin-curvature interactions described by the MPD equations should behave in this way. More generally we believe that the cross section (76) describes how such particles with a non-uniform spatial distribution of spins are scattered by an Einstein-Maxwell pulse in space. In the absence of a relativistic transport description of astrophysical phenomena involving spin-curvature interactions the approach above offers a relatively straightforward mechanism to estimate the significance of such interactions.

## 6 Conclusions

By exploiting the high symmetry of Einstein-Maxwell plane-wave spacetimes we have reduced the pole-dipole MPD equations for the motion of massive particles with spin to a system of tractable ordinary differential equations. Classes of exact solutions have been found corresponding to particular initial conditions for the directions of the particle spin relative to the direction of the propagating background fields. For Einstein-Maxwell pulses we have defined a scattering cross section that reduces in certain limits to those associated with the scattering of scalar and Dirac particles based on classical and quantum field theoretic techniques. Such techniques are considerably more intricate than those discussed here. The relative simplicity of our approach and its use of macroscopic spin distributions suggests that it may have advantages in those astrophysical situations that involve strong classical gravitational and electromagnetic environments.

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